Exact dark energy star solutions

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Abstract

Adopting the phantom (ghost) scalar field description of dark energy, we construct a general class of exact interior solutions describing mixed relativistic stars containing both ordinary matter and dark energy in different proportions. The exterior solution that continuously matches the interior solutions is also found. Exact solutions describing extremal configurations with zero ordinary matter pressure are also constructed.

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The astronomical observations of the present Universe provide evidence for the existence of a mysterious kind of matter called dark energy which governs the expansion of the Universe [1], [2]. The dark energy exhibits some unusual properties such as negative pressure to density ratio w (in hydrodynamical language) and violation of the energy conditions. The ratio w may be less than $-1/3$ corresponding to violation of the strong energy condition, or even less than -1 which is violation of the weak or null energy condition. Current experimental data shows that w is in the range $-1.38 < w < -0.82$.

The fundamental role that the dark energy plays in cosmology naturally makes us search for local astrophysical manifestation of it. In the present work we consider models of relativistic stars containing not only ordinary matter but also dark energy. The existence of dark energy makes us expect that the present-day existing stars are a mixture of both ordinary matter and dark energy in different proportions. The study of such mixed objects is a new interesting problem and some steps in this direction have already been made (see for example [5]- [13] and references therein).

A possible theoretical description of the dark energy is provided by scalar fields with negative kinetic energy, the so-called phantom (ghost) scalars [3], [4]. The negative kinetic energy, however, leads to severe quantum instabilities* and this is a formidable challenge to the theory. However, there are claims that these instabilities can be avoided [14]. In general, the problem could be avoided if we consider the phantom scalars as an effective field theory resulting from some kind of fundamental theory with a positive energy [15], [16]. In this case the phantom scalar description of the dark energy is physically acceptable. In this context it is worth noting that the phantom-type fields arise in string theories and supergravity [19]- [22].

In this work we adopt a description of the dark energy by a phantom scalar. Then the Einstein equations in the presence of dark energy read

$$\begin{aligned} &R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) - 2\partial_\mu\varphi\partial_\nu\varphi, \\ &\nabla_\mu\nabla^\mu\varphi = 4\pi\rho_D. \end{aligned} \tag{1}$$

Here $T_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter in the perfect fluid description with energy density ρ and pressure p:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{2}$$

For the ordinary matter we impose the natural conditions $\rho \geq 0$ and $p \geq 0$. The density of the dark energy sources is denoted by ρ_D. The dark energy sources will be called *dark charges*.

Since a very little is kown about the interaction of dark energy with the normal matter, in our model we consider only minimal interactions for the phantom field – we have not included

*The perfect fluid description of the dark energy also suffers from instabilities due to the imaginary velocity of the sound. From a classical point of view the massless phantom field is even more stable than its usual counterpart [17], [18].

terms describing non-minimal interaction between the phantom field and the normal matter in the field equations (1).

When we consider the local manifestation of dark energy on astrophysical scales, i.e. scales much smaller than the cosmological scales, the phantom potential $\mathcal{U}(\varphi)$ can be neglected and that is why we set $\mathcal{U}(\varphi) = 0$. This is not a principle restriction. Exact solutions with the desired properties as those presented below can be also found in the presence of phantom potential and non-minimal interaction between the ordinary matter and the phantom scalar. Since we do not seek mathematical generality but we are interested in the physics of the model we shall restrict ourselves to the case $\mathcal{U}(\varphi) = 0$ and minimal interaction with the ordinary matter.

In what follows we give exact solutions describing mixed relativistic dark energy stars. The dark star solutions are characterized by the mass M, the dark charge D and the (coordinate) radius R.

We consider static, spherically symmetric and asymptotically flat space-time with a metric

$$ds^2 = -e^{2U}dt^2 + e^{-2U+2\lambda}\left[e^{-2\chi}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{3}$$

where the metric functions are functions of the radial coordinate only. The phantom field φ and the dark charge density ρ_D are required to be static and to depend on the radial coordinate r only. For the normal matter fluid we impose the usual conditions for staticity and spherical symmetry $p = p(r)$, $\rho = \rho(r)$ and $u_\mu dx^\mu = -e^U dt$. Applying the mathematical techniques developed in [23] we can generate exact interior solutions to the field equations (1) using any exact interior solution of the ordinary Einstein-perfect-fluid equations as a seed. More precisely, we have the following proposition:

Proposition *Let*

$$\begin{aligned} ds_E^2 &= -e^{2\lambda}dt^2 + e^{-2\chi}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ p_E &= p_E(r), \\ \rho_E &= \rho_E(r) \end{aligned} \tag{4}$$

be an interior solution to the ordinary Einstein-perfect-fluid equations, then the metric (3) together with the functions

$$\begin{aligned} e^{2U} &= e^{2\lambda\cosh\beta}, \\ \rho &= e^{2U-2\lambda}\left[\rho_E\cosh\beta + 3(\cosh\beta - 1)p_E\right], \\ p &= e^{2U-2\lambda}p_E, \\ \rho_D &= e^{2U-2\lambda}\left(\rho_E + 3p_E\right)\sinh\beta, \\ \varphi &= \sinh\beta\lambda \end{aligned} \tag{5}$$

form an interior solution to the field equations (1) where β is an arbitrary real constant.

It is worth noting that the solution generating method described in the proposition can be applied for an arbitrary equation of state of the ordinary matter.

If the seed solution (4) has a well-defined boundary $r = R$, where by definition $p_E(R) = 0$, the same is true for the solution (5), namely $p(R) = 0$. Therefore R can be interpreted as the coordinate radius of the dark energy star. On the dark star surface $r = R$, our interior solution (5) matches continuously the following exterior solution

$$ds^2_{ext} = -\left(1 - \frac{2m}{r}\right)^{\cosh\beta} dt^2 + \left(1 - \frac{2m}{r}\right)^{1-\cosh\beta} \left[\frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$
$$\varphi_{ext} = \frac{1}{2}\sinh\beta \ln\left(1 - \frac{2m}{r}\right). \tag{6}$$

For $\beta = 0$ this solution reduces to the vacuum Schwarzschild solution describing the exterior gravitational field of a static and spherically symmetric star with mass m. The exterior solution (6) can be easily and elegantly generated from the vacuum Schwarzschild solution if the $SO(1,1)$ symmetry of the dimensionally reduced field equations (1) with $\rho = p = \rho_D = 0$ is used. However, it should be mentioned that (6) was obtained for the first time in [24] (with another method) and can be considered as a phantom counterpart of Fisher's scalar vacuum solution for a normal scalar [25], [26].

The mass and the dark charge of the dark star are given by

$$M = -\frac{1}{4\pi}\int_{Star} R^t_t \sqrt{-g} d^3x = \int_{Star} (\rho + 3p)\sqrt{-g} d^3x = \tag{7}$$
$$\cosh\beta \int_{Star} e^{2\lambda}(\rho_E + 3p_E)\sqrt{-g_E} d^3x = \cosh\beta\, m,$$

$$D = \int_{Star} \rho_D \sqrt{-g} d^3x = \frac{1}{4\pi}\oint_{S^2_\infty} \nabla_\mu \varphi d\Sigma^\mu = \tag{8}$$
$$\sinh\beta \int_{Star} e^{2\lambda}(\rho_E + 3p_E)\sqrt{-g_E} d^3x = \sinh\beta\, m$$

The same expressions for the mass and the dark charge are also obtained from the asymptotic expansion of the exterior solution. The mass M and the dark charge D satisfy the inequality $M > |D|$ as one can see from the above expressions. The dark charge D is a measure of the content of dark energy in the star. When there is no dark energy in the star (i.e. $D = 0$) our solution reduces to the seed interior solution describing ordinary perfect fluid star in general relativity. The initial solution parameters m and β can be expressed in terms of M and D as follows

$$m = \sqrt{M^2 - D^2}, \quad \cosh\beta = \frac{M}{\sqrt{M^2 - D^2}}. \tag{9}$$

In the case of ordinary stars ($D = 0$), the radius of the star R and the gravitational Schwarzschild radius $R_G = 2m$ satisfy the well-known Buchdahl inequality [27]

$$\frac{2m}{R} < \frac{8}{9}. \tag{10}$$

For the mixed dark stars we have

$$\frac{2M}{R_{ph}} = \frac{2m\cosh\beta}{R}\left(1 - \frac{2m}{R}\right)^{(\cosh\beta - 1)/2}, \tag{11}$$

where $R_{ph} = R\left(1 - \frac{2m}{R}\right)^{(1-\cosh\beta)/2}$ is the physical radius of the dark star. By examining the right hand side of eq.(11) and taking into account eq.(10) one can show that for arbitrary β we have

$$\frac{2M}{R_{ph}} < \frac{8}{9}. \tag{12}$$

As an illustration of the above presented method for generating exact dark energy star solutions we will give a fully explicit dark energy star interior solution generated from the well-known interior Schwarzschild solution [28]. The interior Schwarzschild solution qualitatively describes the general case of a static, spherically symmetric perfect fluid star in general relativity and, therefore we expect that the generated solution should qualitatively describe the general case of mixed dark stars. With the generation techniques applied on the interior Schwarzschild solution we obtain:

$$ds^2_{int} = -e^{\frac{2M\lambda}{\sqrt{M^2-D^2}}}dt^2 + e^{-2\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}\lambda}\left[\frac{dr^2}{1 - \frac{2\sqrt{M^2-D^2}}{R^3}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{13}$$

$$p = \frac{3\sqrt{M^2-D^2}}{4\pi R^3}e^{2\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}\lambda}\left[\frac{\left(1 - \frac{2\sqrt{M^2-D^2}}{R^3}r^2\right)^{1/2} - \left(1 - \frac{2\sqrt{M^2-D^2}}{R}\right)^{1/2}}{3\left(1 - \frac{2\sqrt{M^2-D^2}}{R}\right)^{1/2} - \left(1 - \frac{2\sqrt{M^2-D^2}}{R^3}r^2\right)^{1/2}}\right], \tag{14}$$

$$\rho = \frac{3M}{4\pi R^3}e^{2\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}\lambda} + 3\frac{M - \sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}p, \tag{15}$$

$$\rho_D = \frac{3D}{4\pi R^3}e^{2\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}\lambda} + \frac{3D}{\sqrt{M^2-D^2}}p, \tag{16}$$

$$\varphi = \frac{D}{\sqrt{M^2-D^2}}\lambda, \tag{17}$$

where

$$e^{\lambda}=\left[\frac{3}{2}\left(1-\frac{2\sqrt{M^2-D^2}}{R}\right)^{1/2}-\frac{1}{2}\left(1-\frac{2\sqrt{M^2-D^2}}{R^3}r^2\right)^{1/2}\right]. \quad (18)$$

Contrary to the pressure, the fluid energy density ρ and the dark charge density ρ_D do not vanish on the boundary just as in the case of the fluid energy density in the interior Schwarzschild solution. We have

$$\rho(R)=\frac{3M}{4\pi R^3}\left(1-\frac{2\sqrt{M^2-D^2}}{R}\right)^{\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}}, \quad (19)$$

$$\rho_D(R)=\frac{3D}{4\pi R^3}\left(1-\frac{2\sqrt{M^2-D^2}}{R}\right)^{\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}}. \quad (20)$$

Our interior solution is completely regular everywhere for $0\leq r\leq R$ if the dark star radius R satisfies the inequality

$$\frac{\sqrt{M^2-D^2}}{R}<\frac{4}{9}. \quad (21)$$

On the dark star surface $r=R$, our interior solution (13) matches continuously the exterior solution (6) which in terms of M and D reads

$$\begin{aligned} ds^2_{ext} &= -\left(1-\frac{2\sqrt{M^2-D^2}}{r}\right)^{\frac{M}{\sqrt{M^2-D^2}}}dt^2 \\ &+\left(1-\frac{2\sqrt{M^2-D^2}}{r}\right)^{-\frac{M-\sqrt{M^2-D^2}}{\sqrt{M^2-D^2}}}\left[\frac{dr^2}{1-\frac{2\sqrt{M^2-D^2}}{r}}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \quad (22) \\ \varphi_{ext} &= \frac{D}{2\sqrt{M^2-D^2}}\ln\left(1-\frac{2\sqrt{M^2-D^2}}{r}\right). \quad (23) \end{aligned}$$

This solution reduces to the exterior Schwarzschild solutions in the absence of dark energy (i.e. for $D=0$).

We will also consider extremal dark star configurations which can be obtained from (13) by taking the limit $|D|\to M$ and keeping R fixed. The described limit gives the following interior solution

$$\begin{aligned} & ds^2_e=-e^{-\frac{M}{R}\left(3-\frac{r^2}{R^2}\right)}dt^2+e^{\frac{M}{R}\left(3-\frac{r^2}{R^2}\right)}\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \\ & \rho=\frac{M}{\frac{4\pi R^3}{3}}e^{-\frac{M}{R}\left(3-\frac{r^2}{R^2}\right)}, \\ & \rho_D=\pm\rho, \\ & p=0, \\ & \varphi=\mp\frac{M}{2R}\left(3-\frac{r^2}{R^2}\right). \end{aligned} \quad (24)$$

This extremal interior solution matches continuously the exterior solution which is obtained from (22) in the limit $|D| \to M$, namely

$$ds_e^2 = -e^{-\frac{2M}{r}}dt^2 + e^{\frac{2M}{r}}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{25}$$

$$\varphi = \mp\frac{M}{r}. \tag{26}$$

As one can see from (24) the pressure p of the ordinary matter vanishes in the extremal limit. How can one explain the existence of such extremal dark stars with zero pressure? Generally speaking, the phantom field yields repulsive rather than attractive force. From the contracted Bianchi identities

$$\partial_r p + (\rho + p)\partial_r U = \rho_D \partial_r \varphi, \tag{27}$$

describing the equilibrium, one can see that the dark energy term $\rho_D\partial_r\varphi$ provides additional effective pressure which can balance the gravitational forces.

The extremal configurations in the general case can be investigated on the basis of the field equations (1). Following [23] one can show that for extremal configurations with zero pressure ($p = 0$) the 3-metric $h_{ij} = e^{-2U}g_{ij}(i,j = 1,2,3)$ is flat (i.e $h_{ij} = \delta_{ij}$), $\rho_D = \pm\rho$, $\varphi = \pm U$ and that U satisfies the equation

$$\Delta_f U = 4\pi\rho e^{-2U}, \tag{28}$$

where Δ_f is the ordinary flat Laplacian. Solving this equation in vacuum, i.e. for $\rho = 0$, and taking into account that $g_{ij} = \delta_{ij}e^{2U}$ and $\varphi = \pm U$ we obtain the exterior solution (25). Therefore the exterior extremal solution (25) is the same for all extremal configurations with zero ordinary matter pressure .

Exact extremal interior solutions can be found by specifying the dependance $\rho = \rho(U)$. For example the extremal interior solution (24) is obtained for $\rho e^{-2U} = const$. Another exact and physically well behaved solution is found for $\rho e^{-2U} = -C^2/4\pi \times U$ where $C > 0$ is a constant. In this case we have to solve the equation

$$\Delta_f U + C^2 U = 0. \tag{29}$$

The solution of this equations which is well behaved at the center $r = 0$ is

$$U = A\frac{\sin Cr}{r}, \tag{30}$$

with A being an integration constant. In order to match the exterior and interior solution we require that the potential U and its radial derivative are continuous on the boundary $r = R$. Imposing these conditions we obtain the following solution

$$U = -M\frac{\sin\left(\frac{\pi}{2}\frac{r}{R}\right)}{r}. \tag{31}$$

Choosing appropriate functions $\rho = \rho(U)$ one can construct many other extremal solutions.

Finally, we give the upper bound of the ratio $2M/R_{ph}$ for the extremal configurations, namely

$$\frac{2M}{R_{ph}} = \frac{2M}{R} e^{-M/R} \leq \frac{2}{e}. \tag{32}$$

In conclusion, the exact interior solutions (and the exterior solution) found in the present paper could be used in studying, both qualitatively and quantatively, the local astrophysical effects related to the existence of dark energy. Some implications of the solutions presented in the present paper will be considered in a future publication. There remain some issues that need further investigation. One such issue is the stability of the found solutions. The fact that the phantom scalar field exhibits more stable behavior than its canonical counterpart [17], [18] and the results in [13] show that we should expect that the non-extremal solutions are classically stable when the seed Einstein-perfect-fluid solution is stable. This problem will be considered more thoroughly in a future publication.

The present work could also be extended in the following direcrion. Performing an approprate analytical continuatioin of the exterior solution (6) we obtain the well-known Bronnikov–Ellis phantom scalar (vacuum) wormhole solution [29], [30]. It is interesting whether Bronnikov–Ellis branch can also be the external field of a star and what is the structure of such a star. We anticipate that some interior solutions countiniously matching the exterior Bronnikov–Ellis solution could be obtained by an approprate analytical continuatioin of some of the interior solutions constructed in this paper and probbaly some of those solutions would have structure similar to that considered in [13].

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